Chapter 9

t-Motifs with Large Endomorphism Rings

9.1 O_D-Motifs

9.1.1. Fix a finite field k of cardinality q and a field F/k of transcendence degree 1 whose field of constants is k. Choose a place ∞ of F. Pick a function $t \in F$ that has a pole of order prime to p at ∞ and is regular everywhere else. Such a function exists by Riemann-Roch. Let $A \subset F$ be the ring of functions that have poles only at ∞. Then A is a $k[t]$-algebra—the integral closure of $k[t]$ in F.

Denote the degree of F over $k(t)$ by r.

9.1.2. Also, fix a central simple F-algebra D of dimension d^2. If L is an algebraically closed field containing k, then $D \otimes_k L$ is isomorphic to the full $d \times d$ matrix algebra over $F \otimes_k L$ by Tsen's Theorem.(1) We will assume that D is unramified at ∞, that is, that $D \otimes_F F_\infty \approx \mathrm{M}(d, F_\infty)$. Choose a maximal A-order $O_D \subset D$ (the condition that O_D be maximal is not really essential.) The objects that will be parametrised in the following chapters can be thought of as certain classes of 'effective t-motifs with O_D-multiplication'.

(1)*Hauptsatz* of [Tsen 1933].

9.1.3. Let K be a field containing k, equipped with an injective and k-linear homomorphism $A \to K$, and hence *a fortiori* equipped with an injective structure homomorphism $k[t] \to K$.

Definition. An O_D-*motif* is a pair (M, α) consisting of

- an effective t-motif M over K, of rank rd^2 and,
- a homomorphism $\alpha : O_D \to \mathrm{End}(M)$ of $k[t]$-algebras,

such that M is free of rank d over $K[\sigma]$ and such that the induced map

$$O_D \otimes_A K^a \to \mathrm{End}(M_{K^a}/\sigma M_{K^a})$$

is an isomorphism of K^a-algebras. A *morphism* between such objects is an O_D-equivariant map of effective t-motifs.

Some immediate consequences of the definition follow. The actions of t and θ on $M_{K^a}/\sigma M_{K^a}$ coincide. There is a natural structure of left $O_D \otimes_k K$-module on M and likewise $M(t)$ is a left $D \otimes_k K$-module. Over K^a, the top exterior power $\wedge^{rd^2} M$ is isomorphic to C^d.

9.1.4. This definition contains the Drinfeld A-modules of all ranks, albeit indirectly, through the mechanism of Morita equivalence which we recall here.

Let R be any ring and n an integer. Consider the functor

$$M \rightsquigarrow N \stackrel{\text{def}}{=} nM = M \oplus M \oplus \cdots \oplus M$$

from the category of left R-modules to the category of left $\mathrm{M}(n, R)$-modules. Let e be the idempotent matrix whose first entry equals 1 and which is zero in all other entries. Then a two-sided inverse functor is given by $N \rightsquigarrow M \stackrel{\text{def}}{=} eN$ and one says that R and $\mathrm{M}(n, R)$ are *Morita equivalent*.[(2)]

Now if $D = \mathrm{M}(d, F)$ and $O_D = \mathrm{M}(d, A)$ and M an O_D-motif, then as above, $M \approx N^d$ as $A \otimes K$-modules. Since the idempotent e can be taken inside O_D—so that σ and e commute—the module N inherits a

(2) After Kiiti Morita who investigated when two rings have equivalent categories of modules. *See* [Morita 1958].

semilinear action and becomes an effective t-motif of rank one over $K[\sigma]$ carrying an action of A. In short: N is a Drinfeld A-module (see §1.3). It follows that the category of $\mathrm{M}(d, A)$-motifs is equivalent to the category of Drinfeld A-modules of A-rank d.

9.2 Purity & Analytic Triviality

9.2.1. Assume that K is algebraically closed. By Tsen's Theorem there exists a finite field l containing k and contained in K such that $D \otimes_k l$ is isomorphic to $\mathrm{M}(d, F \otimes_k l)$. Choose such an l and denote by n its degree over k.

Lemma. *Let M be an O_D-motif. There exists an $\mathrm{M}(d, A)$-motif M' such that $M(t) \approx M'(t)$ as $K(t)[\sigma^n]$-modules.*

Proof. Choose an isomorphism $D \otimes_k l \to \mathrm{M}(d, F \otimes_k l)$ such that $O_D \otimes_k l$ lands inside $\mathrm{M}(d, A \otimes_k l)$. This is possible since all maximal orders in $\mathrm{M}(d, F \otimes_k l)$ are conjugate. The following commutative diagram summarises the resulting identifications.

$$\begin{array}{ccc} D \otimes_k l & \longrightarrow & \mathrm{M}(d, F \otimes_k l) \\ \uparrow & & \uparrow \\ O_D \otimes_k l & \hookrightarrow & M(d, A \otimes_k l) \end{array}$$

Note that the commutative square is τ^n-equivariant but it is τ-equivariant *only* when $D \approx \mathrm{M}(d, F)$.

Assume first that M is a free $O_D \otimes_k K$-module and let e be a generator. Then the t-motif M is determined by the image of e under σ and thus by the $S \in O_D \otimes_k K$ with $\sigma(e) = Se$. Put $M' \overset{\text{def}}{=} \mathrm{M}(d, A \otimes_k K)f$, the $\mathrm{M}(d, A)$-motif generated by f and with $\sigma(f) \overset{\text{def}}{=} Sf$. Here S acts via the embedding $O_D \otimes_k K \to \mathrm{M}(d, A \otimes_k K)$ induced from the identifications of the commutative square. Clearly $M'(t) \approx M(t)$ as $K(t)[\sigma^n]$-modules.

In general M is not free but locally free (for the Zariski topology on A). It can be embedded in a free module $(O_D \otimes_k K)^m$ for some m. Let $e_1, \ldots, e_s$ be a generating set for $M \subset (O_D \otimes_k K)^m$. Then define

M' as the $\mathrm{M}(d, A \otimes_k K)$-module generated by the images of $e_1, \dots, e_s$ in $\mathrm{M}(d, A \otimes_k K)^m$. By the identifications M' carries a σ-action and by construction $M'(t) \approx M(t)$ as $K(t)[\sigma^n]$-modules. □

9.2.2. $\mathrm{M}(d, A)$-motifs are direct sums of Drinfeld modules and we have already seen that Drinfeld modules are pure (6.2.2) and analytically trivial (3.2.7). Let M be an O_D-motif. The Lemma readily implies:

Corollary. *M is analytically trivial and pure of weight $1/dr$.*

Proof. Let M be an O_D-motif and take M' as in the Lemma.

M' is pure of weight $1/dr$, and it follows immediately from the definition of pureness that M is also pure of the same weight.

As M' is a direct sum of Drinfeld modules, it is analytically trivial. Thus, in particular

$$M(\{t\})^{\sigma^n} \approx M'(\{t\})^{\sigma^n} \approx l(t)^{rd^2}.$$

The action of σ on the left hand side induces a semi-linear action on $l(t)^{rd^2}$ and Hilbert 90 (b.1.1) implies that $M(\{t\})^{\sigma} \approx k(t)^{rd^2}$ and that M is analytically trivial. □