Chapter 10

Moduli

10.1 Periods of O_D-Motifs

10.1.1. We continue with the notation of the previous chapter. Moreover, we assume that $K = K^\dagger$.

Let M be an O_D-motif. Since M is analytically trivial (§9.2), the module $H_{\text{an}}(M, k[t])$ is a left O_D-module which is locally free of rank one. (locally, with respect to the Zariski topology on $\text{Spec}(A)$)

The period construction (§8.2) gives rise to an injective map

$$H_{\text{an}}(M, k[t]) \otimes_{k[t]} K[[t-\theta]] \xrightarrow{\omega_M} M \otimes_{K[t]} K[[t-\theta]] \qquad (10.1)$$

which, by transport of structure, is a map of left $O_D \otimes_{k[t]} K[[t-\theta]]$-modules.

Since O_D-motifs are pure (§9.2) it follows from Theorem 8.3.2 that the map (10.1) determines M.

10.1.2. The above induces an injective map on the dual modules:

$$(M \otimes_{K[t]} K[[t-\theta]])^\vee \xrightarrow{\omega_M^\vee} H_{\text{an}}(M, k[t])^\vee \otimes_{k[t]} K[[t-\theta]] \qquad (10.2)$$

Denote by $\text{Lie}(M)$ the cokernel of $\omega_M^\vee$—this is just a K-vector space. Since the top exterior power of M is isomorphic to C^d the determinant of $\omega_M^\vee$ is $(t-\theta)^d$ up to a unit in $K[[t-\theta]]$. Therefore $\text{Lie}(M)$ is of dimension d. It is equipped with a right action by $O_D \otimes_{k[t]} K$.

The cokernel map restricts to a homomorphism

$$H_{\mathrm{an}}(M, k[t])^{\vee} \to \mathrm{Lie}(M) \tag{10.3}$$

of right O_D-modules, about which we know the following:

Proposition. *The map (10.3) satisfies*

- *the target* $\mathrm{Lie}(M)$ *is a simple right* $O_D \otimes_A K$*-module;*
- *the source* $H_{\mathrm{an}}(M, k[t])^{\vee}$ *is a right* O_D*-module, locally free of rank one;*
- *the map is injective,* O_D*-linear and with discrete image.*

Proof of the Proposition. Only 'simple', 'injective' and 'discrete' constitute new claims. Since the t-motifs involved are finitely generated over $K[\sigma]$, the results of §3 of [Anderson 1986] apply. There it is shown that $\mathrm{Lie}(M)$ is dual to $M/\sigma M$ (hence the 'simple' follows from the definition of an O_D-motif) and that the map in question is injective and with discrete image. □

10.1.3. The map (10.3) determines the map (10.2), hence also (10.1) and M. Also, all such maps arise from some M:

Theorem. *The functor*

$$M \rightsquigarrow \left[H_{\mathrm{an}}(M, k[t])^{\vee} \to \mathrm{Lie}(M)\right]$$

is an anti-equivalence from the category of O_D*-motifs to the category of maps* $\Lambda \to V$ *satisfying the three conditions of the above Proposition, that is,*

- V *is a simple right* $O_D \otimes_A K$*-module;*
- Λ *is a right* O_D*-module, locally free of rank one;*
- *the map is injective,* O_D*-linear and with discrete image.*

Proof. It has already been remarked that the functor is fully faithful (essentially because of Theorem 8.3.2.)

To prove that it is essentially surjective, assume first that O_D is split: $O_D = \mathrm{M}(d, A)$. Then the M in question correspond under Morita equivalence (9.1.4) to (sums of) Drinfeld modules and the Theorem follows immediately from ANDERSON's transcription of DRINFELD's work into the language of t-motifs.[1]

By a Theorem of ANDERSON[2] the question whether $\Lambda \to V$ comes from some O_D-motif depends only on the $k((\theta^{-1}))$-span of Λ. Since D_∞ is isomorphic with $\mathrm{M}(d, F_\infty)$ the general case follows immediately from the split case. $\square$

10.2 Analytic Moduli

10.2.1. We write $\mathbf{A}$ for the ring of finite adèles of A:

$$\mathbf{A} \stackrel{\text{def}}{=} \prod_{v \neq \infty}{}' F_v,$$

and $\hat{A}$ for the subring

$$\hat{A} \stackrel{\text{def}}{=} \prod_{v \neq \infty} A_v.$$

We have the identity $\hat{A} \otimes_A F = \mathbf{A}$.

Abusively, we use the symbol $D^\times$ for both the group of units in D and for the associated algebraic group over F, defined by its functor of points: for all F-algebra's R,

$$D^\times(R) \stackrel{\text{def}}{=} (D \otimes_F R)^\times.$$

The subgroup U of $D^\times(\mathbf{A})$ defined as

$$U \stackrel{\text{def}}{=} (O_D \otimes_A \hat{A})^\times \subset (D \otimes_A \hat{A})^\times = D^\times(\mathbf{A})$$

is compact and open.

Since $\|\theta\| > 1$ and since $K = K^\dagger$ is complete the embedding $A \to K$ extends to an embedding $F_\infty \to K$. We need the Drinfeld symmetric space[3]

$$\Omega^d = \Omega^d_{F_\infty} \stackrel{\text{def}}{=} \mathbf{P}(K^d) - \cup_H H(K)$$

(1) *See* [ANDERSON 1986] and [DRINFEL'D 1974], respectively.
(2) Theorem 6 of [ANDERSON 1986].
(3) *See* [DRINFEL'D 1974].

where H runs through the set of F_∞-rational hyperplanes in $\mathbf{P}(K^d)$. This is a rigid analytic space on which the group $\mathrm{PGL}(d, F_\infty)$ acts naturally.

10.2.2. The choice of an isomorphism of $D \otimes_F F_\infty$ with $\mathrm{M}(d, F_\infty)$ identifies $D^\times$ with a subgroup of $\mathrm{GL}(d, F_\infty)$ and therefore induces an action of $D^\times$ on Ω^d. Following a classical argument we prove:

Proposition. *There is a natural bijection between the set of isomorphism classes of O_D-lattices $\Lambda \subset V$ as in Theorem 10.1.3 and the double coset space*

$$D^\times \backslash \Big(\Omega^d \times D^\times(\mathbf{A}) / U \Big)$$

where the action by $D^\times$ is diagonal.

The double coset space is the disjoint union of a finite number of quotients of Ω^d by discrete group actions, and therefore a smooth rigid analytic space. It is proper if and only if D is a division algebra, and it is a curve if and only if $d = 2$. In particular, when D is a division quaternion algebra, then the double coset space is a proper Mumford curve and its genus is the rank of the abelianisation of the group $O_D^\times$.(4)

Even though we only make statements of a set-theoretic nature, it should be clear that the constructions define a rigid analytic moduli space of O_D-motifs. To make such a statement hard we should treat rigid analytic 'families' of O_D-motifs.(5)

Proof of the Proposition. Start off with a $\Lambda \to V$. There is no loss of generality in assuming that $V = K^d$ on which D acts via the chosen isomorphism $D \otimes_F F_\infty = \mathrm{M}(d, F_\infty)$. Consider the F-span $F\Lambda$ of the lattice. This is a free module of rank 1 over D lying inside V and the choice of a generator marks a point on $\mathbf{P}(V)$ and identifies $F\Lambda$ with D. The marked point lies in $\Omega^d \subset \mathbf{P}(V)$ if and only if Λ is discrete in V. The embedding $\Lambda \subset D$ can be tensored to an embedding $\hat{A}\Lambda \subset D \otimes \mathbf{A}$ and the former can be recovered from the latter as $\Lambda = \hat{A}\Lambda \cap D$. But all locally free $O_D \otimes \hat{A}$-modules are free, consequently the locally free O_D-submodules

(4) *See* [Gerritzen and van der Put 1980].
(5) *See* [Böckle and Hartl 2005] for an example of how this can be done.

$\Lambda \subset D$ of rank one are in bijection with the free rank one $O_D \otimes \hat{A}$-submodules of $D \otimes \mathbf{A}$ and the latter are in bijection with $(D \otimes \mathbf{A})^\times / U$. It remains to mod out by the choice of the generator of $F\Lambda$, that is, by $D^\times$, to establish the desired one-to-one correspondence. □

10.3 Algebraic Moduli

LAUMON, RAPOPORT, and STUHLER have studied moduli of O_D-motifs from a purely algebraic point of view.[(6)] They obtain algebraic varieties classifying O_D-motifs, and studying the cohomology of those they prove the local Langlands correspondence for $\mathrm{GL}(d)$ in equal characteristic p. It is mentioned in *loc. cit.* that those algebraic varieties have a rigid-analytic uniformisation as in 10.2.2, but no attempt is made to prove this.[(7)]

The moduli problems of [LAUMON *et al.* 1993] are not defined in terms of t-motifs, but are cast in a slightly different language. It is remarked[(8)] in *loc. cit.* that there is a relation with t-motifs. In this section, this relation will be made explicit.

10.3.1. The main point is to extend the notion of an O_D-motifs M to something that is defined also over the infinite place of F.

We need to fix a maximal order $\mathcal{D}_\infty \subset D_\infty \overset{\text{def}}{=} D \otimes F_\infty$. Such a maximal order is unique up to conjugation. If we put $\mathcal{D}_v = O_D \otimes_A A_v$ for all places $v \neq \infty$ then the $\mathcal{D}_v$ combine to form a sheaf $\mathcal{F}$ on the algebraic curve with function field F.

(6) *See* [LAUMON *et al.* 1993].

(7) It is mentioned on *p.* 308 as one out of two possibilities for showing an unproved statement used in their proof of local Langlands. The alternative approach proposed is based on an at that time unpublished result of I. BERNSTEIN. It is also remarked (on *p.* 309) that a (at that time) forthcoming paper by SCHNEIDER and STUHLER contains a proof of BERNSTEIN's result. I am not sufficiently versed in Representation Theory to verify that this proof has by now indeed appeared in the litterature. It should of course be stressed that the local Langlands correspondence is by now a corollary to the global correspondence constructed in [LAFFORGUE 2002].

(8) On *p.* 228.

10.3.2. We are now about to define the category $\mathcal{C}$ in which the objects that are parametrised in *loc. cit.* live. The definition is quite a mouthful and depends on the notion of a vector bundle on the non-commutative projective line $\mathbf{P}^1_K(\sigma)$. This notion is defined and explained in the appendices, Section a.3.

The category $\mathcal{C}$ depends on the sheaf of orders $\mathcal{D}$, in other words, depends on both O_D and $\mathcal{D}_\infty$.

Definition. An object $\mathcal{V}$ of $\mathcal{C}$ consists of the data

- a rank d vector bundle (M, W) on $\mathbf{P}^1_K(\sigma)$,
- a homomorphism $\alpha : O_D \to \mathrm{End}_{K[\sigma]}(M)$,
- a homomorphism $\alpha_\infty : \mathcal{D}_\infty \to \mathrm{End}_{K[[\sigma^{-1}]]}(W)$,

which are subject to the conditions that

- (M, α) is an O_D-motif,
- the following square commutes:

$$\begin{array}{ccc} F_\infty \otimes_A O_D & \longrightarrow & \mathrm{End}_{K((\sigma^{-1}))}\left(M \otimes_{K[\sigma]} K((\sigma^{-1}))\right) \\ \| & & \| \\ F_\infty \otimes_{\mathcal{O}_\infty} \mathcal{D}_\infty & \longrightarrow & \mathrm{End}_{K((\sigma^{-1}))}\left(W \otimes_{K[[\sigma^{-1}]]} K((\sigma^{-1}))\right) \end{array}$$

  (the horizontal arrows are induced by α resp. α_∞),
- W is finitely generated as $K[[t^{-1}]]$-module,
- $\sigma^{-rd}W = t^{-1}W$.

A morphism in $\mathcal{C}$ is a morphism of vector bundles that is compatible with α and α_∞.

10.3.3. The 'shift' functors that map $\mathcal{V} = (M, W)$ to

$$\mathcal{V}(n) \stackrel{\text{def}}{=} (M, \sigma^n W)$$

are auto-equivalences of $\mathcal{C}$. They define an action of the group $\mathbf{Z}$ on the category $\mathcal{C}$. A quotient category $\mathbf{Z}\backslash\mathcal{C}$ in which the shifts of an object are identified can be constructed by adding morphisms to $\mathcal{C}$. The objects of $\mathbf{Z}\backslash\mathcal{C}$ are then the objects of $\mathcal{C}$ while the morphisms $\mathcal{V}_1 \to \mathcal{V}_2$ are pairs (f, i) of an integer $i \in \mathbf{Z}$ and a $\mathcal{C}$-morphism $\mathcal{V}_1 \to \mathcal{V}_2(i)$. Composition is defined by the rule

$$(f, i) \circ (g, j) = (f(j) \circ g, i + j).$$

10.3.4. In *loc. cit.* families $\mathcal{V}_S$ defined over arbitrary K-schemes S are considered. After modding out the 'shift' as above, these families of classes of $\mathcal{V}$ form a fibered category which is shown to be a smooth algebraic stack, proper if and only if D is a division algebra. Adding level structures to kill the (finite) automorphism groups of the objects, and thus passing to a finite covering, the stack becomes representable.

10.3.5. Thus, in order to identify the rigid analytic moduli spaces defined in the previous section with the algebraic moduli spaces of *loc. cit.*, it remains to show:

Theorem. *The forgetful functor*

$$\mathcal{V} \rightsquigarrow (M, \alpha)$$

defines an equivalence of $\mathbf{Z}\backslash\mathcal{C}$ *with the category of* O_D*-motifs.*

In particular this means that $\mathbf{Z}\backslash\mathcal{C}$ does not depend on $\mathcal{D}_\infty$.

Proof. It is clear that this forgetful functor on $\mathcal{C}$ factors over $\mathbf{Z}\backslash\mathcal{C}$. A quasi-inverse functor can be constructed as follows.

Assume given an (M, α). Consider the Dieudonné module $M((t^{-1}))$. The action of α makes $M((t^{-1}))$ into a free $D_\infty \otimes_k K$-module of rank one. Moreover, by the purity of M (*see* §9.2) $M((t^{-1}))$ is isomorphic to the sum of d copies of $V_{1/dr}$ (*see* Chapter 6), and using for example the standard basis for $V_{1/dr}$ it is not hard to see that the action of σ makes $M((t^{-1}))$ (with its t^{-1}-adic topology) into a d-dimensional topological vector space over $K((\sigma^{-1}))$ (with its σ^{-1}-adic topology.) Clearly also

$$M((\sigma^{-1})) \stackrel{\text{def}}{=} M \otimes_{K[\sigma]} K((\sigma^{-1}))$$

is a d-dimensional topological $K((\sigma^{-1}))$-vector space. In fact, the natural $K((\sigma^{-1}))$-linear map

$$M((\sigma^{-1})) \to M((t^{-1})) \tag{10.4}$$

(extending the embedding $M \to M((t^{-1}))$) is an isomorphism since it is D-equivariant and the image is non-trivial. Identify source and target of this isomorphism.

In order to complete M to an object (M, W) of $\mathcal{C}$, it remains to find a

$$W \subset M((\sigma^{-1})) = M((t^{-1}))$$

that is simultaneously a $K[[\sigma^{-1}]]$-lattice and a $\mathcal{D}_\infty \otimes_k K$-lattice.

The possible W can be determined using Morita equivalence. Choose a primitive idempotent e in the full matrix algebra $\mathcal{D}_\infty$. Then given a lattice W the projection

$$W' \stackrel{\text{def}}{=} eW \subset eM((\sigma^{-1})) = eM((t^{-1})) \approx V_{1/rd}$$

is a rank one $K[[\sigma^{-1}]]$-lattice and an O_{F_∞}-lattice of rank d. Conversely, any such W' induces a W with the desired properties. Choose any $K[[t^{-1}]]$-lattice $W' \subset eM((\sigma^{-1}))$. A direct verification shows that it satisfies the required properties. Clearly, any two choices of W' differ by a power of σ, that is, by a shift in $\mathcal{C}$. □