

Chapter 1

Effective t -Motifs: a Motivated Definition

1.1 Galois Representations

1.1.1. Let K be a field and K^s a separable closure of K . With a smooth and projective algebraic variety X over K one associates ℓ -adic cohomology groups,

$$X \rightsquigarrow H_{\text{ét}}^{\bullet}(X_{K^s}, \mathbf{Q}_{\ell}) = H_{\text{ét}}^{\bullet}(X_{K^s}, \mathbf{Z}_{\ell}) \otimes_{\mathbf{Z}} \mathbf{Q},$$

for every prime number ℓ coprime to the characteristic of K . These are finite dimensional \mathbf{Q}_{ℓ} -vector spaces equipped with a continuous action of the absolute Galois group $G_K \stackrel{\text{def}}{=} \text{Gal}(K^s/K)$. One knows that a number of invariants of these cohomology groups are independent of ℓ —dimension, characteristic polynomials of Frobenius—yet there is no direct way of relating these various cohomologies.

GROTHENDIECK has had the idea that the different ℓ -adic cohomology theories could be mere manifestations of a more profound cohomology with \mathbf{Q} -coefficients. He conjectured a factorisation of the above functor as

$$X \rightsquigarrow h(X, \mathbf{Q}) \rightsquigarrow H_{\text{ét}}^{\bullet}(X_{K^s}, \mathbf{Q}_{\ell}) \tag{1.1}$$

through an object $h(X, \mathbf{Q})$, independent of ℓ , which he baptised the *motif* of X . This object is conjecturally a finite dimensional \mathbf{Q} -linear represen-

tation of some universal affine group scheme over \mathbf{Q} , at least when K is of characteristic zero.⁽¹⁾

1.1.2. Now let k be a finite field of q elements and K any field containing k . Representations of G_K with coefficients in k are considerably more accessible than representations with characteristic zero coefficients, largely because of this:

Theorem. *The following two categories are equivalent.*

- Pairs (H, ρ) of a finite dimensional k -vector space H and a continuous homomorphism $\rho : G_K \rightarrow \mathrm{GL}(H)$,
- Pairs (V, σ) of a finite dimensional K -vector space V and an additive map $\sigma : V \rightarrow V$ satisfying $\sigma(xv) = x^q v$ and such that $K\sigma(V) = V$.

An equivalence is given by the mutually inverse functors

- $H \rightsquigarrow V(H) \stackrel{\text{def}}{=} (H \otimes_k K^s)^{G_K}$ (invariants under G_K), and,
- $V \rightsquigarrow H(V) \stackrel{\text{def}}{=} (V \otimes_K K^s)^\sigma$ (invariants under σ).

Note that if K is not perfect $K\sigma(V)$ need not coincide with $\sigma(V)$, as can be seen already when $(V, \sigma) = (K, x \mapsto x^q)$.

The Theorem can be read as a characteristic p Riemann-Hilbert correspondence: σ has the traits of a connexion on the ‘bundle’ V , while ρ makes H into a local system of k -vector spaces for the étale topology on $\mathrm{Spec} K$.

Proof of the Theorem.⁽²⁾ Consider a pair (H, ρ) . The profinite group G_K acts continuously on the K^s -vector space $H \otimes_k K^s$ through the simultaneous action on both factors of the tensor product. This space has an invariant K^s -basis by ‘Hilbert 90’ for $\mathrm{GL}(n)$ (see the appendices, b.1.1). Thus the map

$$(H \otimes_k K^s)^{G_K} \otimes_K K^s \rightarrow H \otimes_k K^s$$

defined by

$$(h \otimes x) \otimes y \mapsto h \otimes xy$$

⁽¹⁾For background on GROTHENDIECK’S theory of motifs, the reader is advised to consult the lecture [SERRE 1991] or the monograph [ANDRÉ 2004].

⁽²⁾See also Proposition 4.1 of [PINK AND TRAUlsen 2006].

is an isomorphism. By construction it is equivariant under the action of σ and taking invariants one obtains an isomorphism:

$$H(V(H)) = ((H \otimes_k K^s)^{G_K} \otimes_K K^s)^\sigma \approx (H \otimes_k K^s)^\sigma = H.$$

Now start with a (V, σ) . The action of σ on extends to an action on the K^s -space $V \otimes_K K^s$ by $\sigma(v \otimes x) = \sigma(v) \otimes x^q$. By LANG's 'Hilbert 90' Theorem (see b.2.1) $V \otimes_K K^s$ has a σ -invariant basis and it follows that the map

$$(V \otimes_K K^s)^\sigma \otimes_k K^s \rightarrow V \otimes_K K^s : (v \otimes x) \otimes y \mapsto yxv$$

is an isomorphism. It is G_K -equivariant and taking invariants yields the isomorphism

$$V(H(V)) = ((V \otimes_K K^s)^\sigma \otimes_k K^s)^{G_K} \approx (V \otimes_K K^s)^{G_K} = V.$$

Thus $V(-)$ and $H(-)$ are mutually inverse equivalences. \square

1.1.3. Assume given a free and finitely generated $K[t]$ -module M together with a $k[t]$ -linear map $\sigma : M \rightarrow M$ satisfying $\sigma(xm) = x^q m$ for all $x \in K$. Assume also that the K -vector space $K\sigma(M)$ is of finite codimension in M .

Fix an irreducible monic polynomial $\lambda \in k[t]$ and consider $M/\lambda^n M$. This is a finite dimensional K -vector space and since λ and σ commute, the semi-linear action of σ on M carries over to an action on the quotient. For all but finitely many 'bad' λ the resulting action on the quotient $M/\lambda^n M$ satisfies $K\sigma(M/\lambda^n M) = M/\lambda^n M$. For a 'good' λ (and all n) one can thus apply the functor H of the Theorem. The resulting $H(M/\lambda^n M)$ becomes a $k[t]/\lambda^n k[t]$ -module by transport of structure and taking limits yields a functor

$$M \rightsquigarrow H_\lambda(M) \stackrel{\text{def}}{=} \left(\varprojlim_n H(M/\lambda^n M) \right) \otimes_{k[t]} k(t) \quad (1.2)$$

that associates with the pair (M, σ) a continuous representation of G_K on a $k(t)_\lambda$ -module of finite dimension equal to $\text{rk}_{K[t]} M$. It is continuous for

the λ -adic topology since for every n the representation on $H(M/\lambda^n M)$ is continuous.

It is tempting to interpret the collection of functors (1.2)—one for every good place λ —as an analogue to the second functor in (1.1) and we will see in the following chapters that there are in fact many similarities between the modules (M, σ) and the conjectural motifs $h(X, \mathbf{Q})$.

1.1.4. Now let again K be an arbitrary field and ℓ a prime number different from the characteristic of K . Denote by $\mu_{\ell^n}(K^s)$ the set of ℓ^n -th roots of unity in K^s . The ring $\mathbf{Z}/\ell^n\mathbf{Z}$ acts naturally on $\mu_{\ell^n}(K^s)$ via $(m + \ell^n\mathbf{Z}, \zeta) \mapsto \zeta^m$. This action commutes with the obvious action of G_K . Taking the projective limit over n one thus obtains a continuous representation of G_K on the rank one \mathbf{Z}_ℓ -module $\mu_{\ell^\infty} \stackrel{\text{def}}{=} \varprojlim \mu_{\ell^n}$. This ℓ -adic representation is denoted by $\mathbf{Z}_\ell(1)$, and after extension of scalars by $\mathbf{Q}_\ell(1) \stackrel{\text{def}}{=} \mathbf{Z}_\ell(1) \otimes_{\mathbf{Z}} \mathbf{Q}$. Furthermore, representations $\mathbf{Q}_\ell(i)$ are defined for all integers i as $\mathbf{Q}_\ell(i) \stackrel{\text{def}}{=} \mathbf{Q}_\ell(1)^{\otimes i}$.

The representations $\mathbf{Q}_\ell(i)$ are intimately related to class field theory. For example, when $K = \mathbf{Q}$, the following property characterises $\mathbf{Q}_\ell(i)$:

Proposition. *Let $p \neq \ell$ be a prime number, then $\mathbf{Q}_\ell(i)$ is unramified at p and a Frobenius element $g_p \in G_{\mathbf{Q}}$ at p acts on $\mathbf{Q}_\ell(i)$ as multiplication with p^i . \square*

The $\mathbf{Q}_\ell(i)$ with $i \leq 0$ occur inside various ℓ -adic cohomology groups. One has for example:

$$H_{\text{ét}}^{2d}(\mathbf{P}_{K^s}^d, \mathbf{Q}_\ell) = \mathbf{Q}_\ell(-d).$$

The representation $\mathbf{Q}_\ell(i)$ with i positive, however, cannot occur as a piece of the ℓ -adic cohomology of some smooth and projective variety over, say, a number field, since on these groups the traces of geometric Frobenius—inverse to the arithmetic Frobenius of the above Proposition—must be algebraic integers.⁽³⁾

It is most important to note that the prime p plays two very distinct roles in the Proposition. It is staged as a prime of the base field $K = \mathbf{Q}$,

⁽³⁾By a Theorem of DELIGNE (the ‘Weil Conjectures’), see [DELIGNE 1974].

in the guise of the Frobenius g_p , and it features as an element of the coefficient field \mathbf{Q}_ℓ , as multiplication with p^i .

1.1.5. Now put $K = k(\theta)$, the function field of the projective line over the finite field k . A prime of K is either ‘infinity’ or a monic irreducible polynomial $f \in k[\theta]$, which we shall call a finite prime. In contrast with the double role played by p in the previous paragraph, here primes of the base K will be polynomials in θ , while elements of the coefficient fields $k(t)_\lambda$ will be (germs of) functions in t .

Denote by $C = (C, \sigma)$ the pair

$$C \stackrel{\text{def}}{=} K[t]e \text{ with } \sigma(fe) \stackrel{\text{def}}{=} \tau(f)(t - \theta)e,$$

where $\tau(f)$ is the polynomial in $K[t]$ obtained by raising all coefficients of f to the q -th power. The following property of C can be shown to characterise C amongst those (M, σ) with M of rank one. It is essentially due to CARLITZ.⁽⁴⁾

Proposition. *Let $\lambda \in k[t]$ be monic and irreducible. If $f = f(\theta) \in k[\theta]$ is a finite prime of K with $f(t) \neq \lambda$ then $H_\lambda(C)$ is unramified at f and a Frobenius element $g_f \in G_K$ at f acts by multiplication with $f(t)^{-1}$.*

Proof. Denote the degree of f by d . The representation $H_\lambda(C)$ is constructed from the $H(C/\lambda^n C)$. Writing out their definition gives

$$\begin{aligned} H(C/\lambda^n C) &= (K^s[t]e/\lambda^n K^s[t]e)^\sigma \\ &= \{x \in K^s[t]/\lambda^n K^s[t] \mid \tau(x)(t - \theta) = x\}. \end{aligned}$$

Iterate the action of τ on an x in this set and obtain

$$\tau^d(x)(t - \theta)(t - \theta^q) \cdots (t - \theta^{q^{d-1}}) = x.$$

The defining property of the Frobenius element g_f is that it equals τ^d after reduction modulo $f(\theta)$. Moreover, it acts on the $k[t]$ -module $H(C/\lambda^n C)$ by endomorphisms. Thus, in order to conclude it suffices to observe that

$$f(t) = (t - \theta)(t - \theta^q) \cdots (t - \theta^{q^{d-1}})$$

as polynomials in t over the finite field $k[\theta]/f(\theta)k[\theta]$. □

⁽⁴⁾See [CARLITZ 1935].

Define an action of σ on the tensor product $C \otimes C$ diagonally:

$$C \otimes C \stackrel{\text{def}}{=} C \otimes_{K[t]} C \text{ with } \sigma(a \otimes b) \stackrel{\text{def}}{=} \sigma(a) \otimes \sigma(b),$$

and similarly for the higher tensor powers C^i (the number of factors i being positive).⁽⁵⁾ The previous Proposition generalises in a straightforward manner. Take $\lambda \neq f(t)$, then

Proposition. g_f acts on $H_\lambda(C^i)$ as multiplication with $f(t)^{-i}$. □

1.2 Effective t -Motifs

1.2.1. As before, k is a field of q elements and K a field containing k . Fix a homomorphism $k[t] \rightarrow K$ of k -algebras and denote the image of t by θ . This homomorphism is to play the role of the canonical homomorphism from \mathbf{Z} into an arbitrary field. Its kernel replaces the notion of the characteristic of a field— K itself is of course always of positive characteristic p . We shall frequently refer to ‘the field K ’, this is silently understood to contain the structure homomorphism $k[t] \rightarrow K$.

Denote by τ the endomorphism of $K[t]$ determined by $\tau(x) = x^q$ for all $x \in K$ and $\tau(t) = t$.

The following definition goes back to [ANDERSON 1986], although here a slightly less restrictive form is used.

Definition. An *effective t -motif* of rank r over K is a pair $M = (M, \sigma)$ consisting of

- a free and finitely generated $K[t]$ -module M of rank r , and,
- a map $\sigma : M \rightarrow M$ satisfying $\sigma(fm) = \tau(f)\sigma(m)$ for all $f \in K[t]$ and $m \in M$,

such that the determinant of σ with respect to some (and hence any) $K[t]$ -basis of M is a power of $t - \theta$ up to a unit in K .

⁽⁵⁾We use the exponential notation for tensor products ($V^2 \stackrel{\text{def}}{=} V \otimes V$) and the multiplicative notation for direct sums ($2V \stackrel{\text{def}}{=} V \oplus V$).

The condition on the determinant guarantees that the Galois representation $H_\lambda(M)$ is well-defined for all but at most one place λ , the possible exception being the kernel of $k[t] \rightarrow K$.

A morphism of effective t -motifs is a morphism of $K[t]$ -modules making the obvious square commute. The group of morphisms is denoted $\text{Hom}_\sigma(M_1, M_2)$. The resulting category of effective t -motifs over K is denoted by $t\mathcal{M}_{\text{eff}}(K)$ or simply by $t\mathcal{M}_{\text{eff}}$.

1.2.2. For any field $k[t] \rightarrow K$, the pair

$$C \stackrel{\text{def}}{=} K[t]e \text{ with } \sigma(fe) \stackrel{\text{def}}{=} \tau(f)(t - \theta)e$$

that we already encountered is an effective t -motif and we will call it the *Carlitz t -motif*.

1.2.3. Define the *tensor product* of two effective t -motifs as

$$M_1 \otimes M_2 \stackrel{\text{def}}{=} M_1 \otimes_{K[t]} M_2 \text{ with } \sigma(m_1 \otimes m_2) \stackrel{\text{def}}{=} \sigma(m_1) \otimes \sigma(m_2).$$

This is again an effective t -motif.

The pair $(K[t], \tau)$ is an effective t -motif which we shall denote $\mathbf{1}$. We call it the *unit t -motif*, since for every M , one has natural isomorphisms $M \otimes \mathbf{1} = M$ and $\mathbf{1} \otimes M = M$.

1.2.4. If M_1 and M_2 are effective t -motifs then $\text{Hom}(M_1, M_2)$ is naturally a $k[t]$ -module.

Proposition. $\text{Hom}(M_1, M_2)$ is free and finitely generated over $k[t]$.

Proof. ⁽⁶⁾ It is sufficient to show that the natural map

$$\text{Hom}_\sigma(M_1, M_2) \otimes_k K \rightarrow \text{Hom}_{K[t]}(M_1, M_2)$$

is injective. Assume that this is not the case, that is, that there exist k -linearly independent $f_i \in \text{Hom}_\sigma(M_1, M_2)$ and scalars $x_i \in K$ with

$$f_0 + x_1 f_1 + \cdots + x_n f_n = 0 \text{ in } \text{Hom}_{K[t]}(M_1, M_2). \quad (1.3)$$

⁽⁶⁾See Theorem 2 of [ANDERSON 1986].

and assume n to be minimal. The f_i are σ -equivariant, thus

$$f_0(\sigma m) + x_1^q f_1(\sigma m) + \cdots + x_n^q f_n(\sigma m) = 0 \quad \forall m \in M_1.$$

and since $\sigma M_1 \subset M_1$ is free and of finite codimension

$$f_0 + x_1^q f_1 + \cdots + x_n^q f_n = 0 \quad \text{in } \text{Hom}_{K[t]}(M_1, M_2). \quad (1.4)$$

Subtract (1.4) from (1.3) to obtain

$$(x_1 - x_1^q) f_1 + \cdots + (x_n - x_n^q) f_n = 0$$

and deduce from the minimality of n that all coefficients vanish, thus that the x_i are in k , which contradicts the independence of the f_i . \square

1.2.5. To write down an effective t -motif of rank r , choose a $K[t]$ -basis $e = (e_i)$ of M and express the action of σ as $\sigma(e) = Se$ for some $S \in \text{M}(r \times r, K[t])$ whose determinant equals $x(t - \theta)^d$ with $x \in K^\times$ and d a non-negative integer. Every such matrix S determines an effective t -motif.

Given an M_1 of rank r_1 described on a basis by a matrix S_1 as above, as well as M_2 , r_2 and S_2 , we have

$$\text{Hom}_\sigma(M_1, M_2) = \{F \in \text{M}(r_2 \times r_1, K[t]) \mid FS_1 = S_2\tau(F)\}$$

and in particular, two matrices S_1 and S_2 determine the same effective t -motif if and only if $r_1 = r_2 = r$ and there exists an $F \in \text{GL}(r, K[t])$ such that $S_1 = F^{-1}S_2\tau(F)$.

1.2.6. Denote by $K[\sigma]$ the ring whose elements are polynomial expressions of the form

$$x_0 + x_1\sigma + \cdots + x_n\sigma^n$$

and where multiplication is determined by the rule

$$\sigma x = x^q \sigma.$$

Effective t -motifs are naturally left $K[\sigma]$ -modules.⁽⁷⁾

⁽⁷⁾More on the structure of the ring $K[\sigma]$ is in Appendix a.

An effective t -motif may or may not be finitely generated over $K[\sigma]$. For instance: C is but $\mathbf{1}$ is not. When an effective t -motif M is finitely generated over $K[\sigma]$ then it is automatically free.⁽⁸⁾

In that case one can equip M with a free $K[\sigma]$ -basis $f = (f_j)$ and express the action of t by $tf = Tf$ for some $T \in M(d \times d, K[\sigma])$. A matrix $T \in M(d \times d, K[\sigma])$ determines an effective t -motif if and only if $T \equiv \theta I_d + N$ modulo σ , where $N \in M(d \times d, K)$ is a nilpotent matrix. To recover the rank of M from T one cannot just take the determinant of T since $K[\sigma]$ is a non-commutative ring, but the degree of the Dieudonné determinant is well-defined and one has $\text{rk}(M) = \text{deg det } T$ (see Appendix a).

Given two such effective t -motifs M_1, M_2 , equipped with $K[\sigma]$ -bases on which the action of t is expressed by T_1 and T_2 respectively, we have

$$\text{Hom}_\sigma(M_1, M_2) = \{G \in M(d_2 \times d_1, K[\sigma]) \mid GT_1 = T_2G\}$$

and in particular, M_1 and M_2 are isomorphic if and only if $d_1 = d_2 = d$ and there exists a $G \in \text{GL}(d, K[\sigma])$ such that $T_1 = G^{-1}T_2G$.

1.3 Example: Drinfeld Modules

1.3.1. Let M be an effective t -motif that is free of rank 1 as $K[\sigma]$ -module. Denote by Z the centre of the endomorphism ring of M . Thus Z is a $k[t]$ -algebra of finite rank. The following is shown in [DRINFEL'D 1974]:

Theorem. *There exists a projective curve X over k and a closed point $\infty \in X$ such that $Z \approx H^0(X - \infty, \mathcal{O}_X)$. If moreover $k[t] \rightarrow K$ is injective, then $\text{End}(M) = Z$. \square*

1.3.2. Next, fix a $k[t]$ -algebra A such that the spectrum of A is $X - \infty$ for some *smooth* projective curve X and a closed point $\infty \in X$. We call a

⁽⁸⁾Lemma 1.4.5 of [ANDERSON 1986]. Note that in ANDERSON's paper only modules that are finitely generated over $K[\sigma]$ are considered. In fact, what ANDERSON calls a 't-motive' is in our language an 'effective t -motif that is finitely generated over $K[\sigma]$.'

Drinfeld A -module⁽⁹⁾ an effective t -motif M that is free of rank one over $K[\sigma]$ together with an injective homomorphism $A \rightarrow \text{End}(M)$. Such an M is a projective and finitely generated $A \otimes K$ -module. The rank of M over $A \otimes K$ is called (abusively) the A -rank of M . The Carlitz motif C is a Drinfeld $k[t]$ -module of $k[t]$ -rank 1, but its higher tensor powers are not Drinfeld modules. Also shown in *loc. cit.* is:

Theorem. *Over a separably closed field K , Drinfeld A -modules of arbitrary A -rank exist.* □

⁽⁹⁾Introduced in [DRINFEL'D 1974], where the term 'elliptic module' is used. Warning: one usually studies Drinfeld modules in the category *opposite* to the category of effective t -motifs.