

SUMMARY

‘Elke positieve actie begint met critiek’ (‘Criticism is the start of all positive action’)

Hans Freudenthal and the didactics of mathematics

Hans Freudenthal (1905–1990), professor in mathematics, scientist, literator, but above all mathematics-educator, was inextricably linked to the changes which took place in mathematics education and didactics during the second half of the last century. To this day he is the dominating figure in mathematics education in the Netherlands and this is one of the main reasons for dedicating a study like the present to Freudenthal. However, for a professor in mathematics it was anything but obvious to devote himself to mathematics education and didactics and it is interesting to see what made Freudenthal choose this path. But there is more.

This study provides a historical analysis of Freudenthal’s didactic ideas and his didactic career and is partly biographical since it concerns the analysis of Freudenthal as an educator and didactician. It also contributes to the historiography of mathematics education. In an indirect way, it also addresses other, closely related questions such as what is mathematics and where does it start? Which role does mathematics play in society and what influence does it have on the prevailing views concerning its accompanying didactics? His diversity as a scientist and his inexhaustible efforts to establish the didactics of mathematics as a seriously pursued science, made Freudenthal’s influence in this area considerable.

He foresaw an essential, practical role for mathematics in everyone’s life. As a starting point in their education he encouraged students to discover and create mathematics themselves, instead of imposing a ready-made mathematical system on them. In the seventies this point of view was put into practice by the co-workers of the rowo (Institute for the development of mathematics education). The mathematics education thus developed would gain world fame in the following decades.

Today, in the light of the discussions about recent mathematics education, in which the call for ‘genuine’ mathematics instead of the so-called ‘kindergarten’-mathematics can be heard, Freudenthal’s approach seems to be passé. However, the outcome of this study (which is mainly based on documents from Freudenthal’s vast personal archive) shows a more refined picture. The direct identification of ‘kindergarten’-mathematics with Freudenthal’s view on mathematics education

does not seem justified. ‘Realistic mathematics’ as advocated by Freudenthal includes more than a practical introductory. For him school mathematics, among other things, should also aim at teaching ‘genuine’ mathematics in the end. This attributes to the justification of a profound analysis and a renewed introduction to Freudenthal’s ideas.

A roughly chronological structure has been chosen. An introductory chapter on the education of mathematics in the interbellum period (see Chapter II) is followed by a biographical chapter (see Chapter III) outlining Freudenthal’s life. In 1930 the young, promising mathematician Freudenthal moved from Germany to Holland. At Amsterdam University an appointment as assistant of the mathematician L.E.J. Brouwer awaited him. Freudenthal married Susanna Lutter in 1932 and until 1946 they and their four children lived in Amsterdam.

Didactics and education did not take a prominent place among Freudenthal’s activities in the thirties. He spent most of his time on his mathematics and on his work for *Compositio Mathematica*, a journal published by Brouwer. He did, however, experiment with teaching a colloquium on mathematics education for mathematics students. He was not a gifted teacher and most of his students had difficulty keeping up with his classes.

After the occupation of the Netherlands during the Second World War, Freudenthal, being Jewish, was removed from office. This involuntary passivity gave him the opportunity to concentrate on his – until then – slumbering interests. His non-Jewish wife still had access to all libraries, which enabled Freudenthal to proceed working at home. Apart from his mathematics, which he kept doing as best as he could, he concentrated on writing plays, novels and poetry and spent much time with his children.

The arithmetic lessons he taught to his sons became a practical reason to take up studying literature on mathematics education and didactics. Early 1944 he started working on what initially was to be a book i.e. ‘Rekendidaktiek’ (Didactics of Arithmetic, see Chapter IV).

When Freudenthal started writing ‘Rekendidaktiek’, he already envisioned the outline and contents of the book. It should become a didactical textbook, based on scientific experiments and research, without any false ‘scientific’ pretence which he had so often come across in the works of others.

The larger part of the manuscript is taken up with Freudenthal’s criticism of the use of what he called the ‘assisting sciences’. Philosophy, psychology, pedagogy, general didactics, all these ‘assisting sciences’ were rejected by him for various reasons. In doing so he also criticised a large number of arithmetic teaching methods. After all, he wrote in this manuscript, ‘criticism is the start of all positive action’. A motto to which he remained faithful for the rest of his life.

Yet, ‘Rekendidaktiek’ was more than just criticism. While rejecting the various ‘assisting sciences’ one by one, and describing the reasons for this, he also tried to find a way in which these sciences could become useful for the didactics of arith-

metic. He emphasized, for instance, the correct use of statistical methods and the importance of making observations. Describing the conversations he had with his children was a phenomenon he would also use in his later work. Finally, he made a stand against the classical education which was common in these days.

In the final ten pages that have been preserved of the manuscript Freudenthal denounced arithmetic education as a means of teaching to think logically – something that was viewed as the main goal of arithmetic education in many arithmetic teaching methods. According to him, ‘learning to think logically’ could be much more efficiently acquired through the learning of the native language. The structure of ‘arithmetic thinking’ and the thinking of formal logic were too simple, too transparent and therefore not typical of thinking in general. According to him arithmetic teaching did not have any other important practical implication besides ‘arithmetic thinking’.

The manuscript ‘Rekendidaktiek’, however, remained unfinished. After the liberation in 1945, Freudenthal resumed his activities at the university and let arithmetic education be. In 1946 he was appointed professor in pure and applied mathematics at the university of Utrecht, where he would live and work the rest of his life. There, an enormous amount of work establishing the Mathematical Institute awaited Freudenthal. That his little excursion to the didactics of arithmetic would remain just that, would have seemed obvious. Nothing was further from the truth.

After the liberation of the Netherlands (1945), Freudenthal seemed to be bursting with impatience to make his pedagogical-didactical findings from wartime known (see Chapter v). At first, on his own initiative he chose the innovative audience of the wvo, the Working community for education and teaching. In August 1945, the wvo organised a summer conference, where Freudenthal spoke about ‘pedagogy of arithmetic and mathematics’, as well as ‘educating for thinking in the broadest sense’. His lecture ‘Educating for thinking’, in which many themes from ‘Rekendidaktiek’ could be recognized, was – according to himself – very successfully received. This was not surprising, as his lecture fitted very well in wvo-circles where conceptions such as ‘education is accompanying the child in its natural growth’ and ‘educating for peace’ were commonly accepted. Freudenthal’s lecture did not merely concern didactics (of arithmetic). He mostly spoke about educating and teaching the masses to think. By the way, he did not think to accomplish this through arithmetic education, but by stimulating reading because, as he wrote, ‘a child that reads twice as much, learns to think twice as much’.

His own contribution to this ‘educating through reading’ were articles published in the weekly journal *De Groene Amsterdammer*. He himself had initiated this connection by sending his article ‘Calculating machines win the war’ to the editors, who published it the following month. It would be the first of a long series of articles he was going to write. Easy to read articles about mathematical and scientific issues, about academic innovations and education in general. In the seventies and eighties he would find a way to vent his literary (and other) comments by writing columns for the daily paper *NRC/Handelsblad*.

After the war, Freudenthal's commitment to carry out his ideas about didactics and education were unstoppable. These efforts did not remain without consequence. His speaking at the wvo-conference put him in contact with the 'Wiskunde Werkgroep' (Mathematics Study Group) of the wvo. In this ww-group he found not only a sounding board but also inspiration for his ideas concerning mathematics didactics, working together with mathematicians and physicists such as Wansink, Ehrenfest-Afanassjewa and the married couple Van Hiele. He defined the ww for himself as an 'environment for gaining mathematics education consciousness'. In 1950 he became president of the ww and it did not take long before Freudenthal was completely involved in the current debates about the modernization of mathematics education at the secondary schools. In doing so, he was not afraid to take a stand against prevailing opinions, like those concerning the formative value of mathematics education.

In the interbellum period the formative value of mathematics was seen as one of the greatest virtues of mathematics education. Teaching mathematics as a deductive, closed system would do optimal justice to its formative value. In this way, mathematics was presented to the student as a ready-made subject. Advocates of this approach, such as H.J.E. Beth and E.J. Dijksterhuis, thought that purity and honesty of mathematical thinking were to be of the utmost importance for the intellectual development and the lifting of the moral level of the student. With this approach, school mathematics would acquire a rather elite character as it meant that understanding mathematics was reserved only for a lucky few.

Freudenthal renounced this formative value of mathematics. In his opinion mathematics education should foremost aim at the teaching of mathematics, and he considered this difficult enough. To develop thinking abilities, students should learn to weigh arguments against each other in an abundance of pros and cons. Mathematics, with its straightforward reasoning and well-organized structures, did not contribute anything to such thinking abilities.

Freudenthal did not mince his words in his publications. 'Abolish this nonsense-science!' he wrote in an article on mechanics education at secondary schools. With these subtlety-lacking utterances he did not make things easy for himself. But still, his opinion was valued. The regards he received as a professor in mathematics contributed to that, and since the prevailing opinions on the education of mathematics were slowly changing, there was more room for modernization.

In 1958 a new mathematics curriculum was adopted, which was partly based on extensive preliminary work of the ww. The sharp distinction between the 'innovators' of the ww and the 'conservatives' of the teachers associations disappeared. In the fifties, Freudenthal's position shifted (see Chapter VI). From a critical outsider he became a true authority in the field of mathematics education and didactics, not in the least due to his increasing flow of publications on these subjects. With unflagging zeal he remained critical, using every opportunity to emphasize the abuses in Dutch mathematics teaching. The old-fashioned curriculum, the abominable final exams, the lack of a genuine teachers' training and the huge gap that existed be-

tween the caricature of mathematics that students learned at high school and the mathematics they came to know at university.

With his reputation as a mathematician (one who also made out a case for all issues related to pedagogy, didactics and education) established, Freudenthal appeared more frequently on the international scene. He became chairman of the 'NOCW' the Dutch committee for mathematics education, and hence the Dutch representative of 'International Commission on Mathematical Instruction' (ICMI). In 1966 he became president of the ICMI and in 1968 he founded the international journal for mathematics education, *Educational Studies in Mathematics*, of which he was general editor.

The writing of the doctoral theses of the Van Hieles meant a turning point in the development of Freudenthal's ideas (see Chapter VII). Freudenthal acted as supervisor for Pierre van Hiele, but was also indirectly involved with Dieke van Hiele-Geldof's doctoral thesis. The main influence of the work by the Van Hieles on Freudenthal's didactical ideas was not so much found in its direct contents, as well as in the development of the theory it initiated for him. Up until then, his publications on education contained few to no descriptions of concrete educational matters such as micro-didactics or learning processes. He had mainly pointed out abuses and criticized what were known as 'traditional' education methods. He had not gone much further than indicating the direction in which should be looked for improvement and solutions, thereby wholeheartedly admitting that he could not do much more either, as long as decent research results were missing.

But this changed with the work of the Van Hieles. Freudenthal highly appreciated and admired the analysis of classroom observations of Mrs. Van Hiele and the intuitive approach she promoted in introductory geometry education. However, it was Mr. Van Hiele's 'theory of levels' that would mostly influence Freudenthal's ideas. In his theory of levels Van Hiele, inspired (among others) by Piaget, described the different levels students had to go through in their learning processes in geometry education and the leaps occurring between the subsequent levels of understanding. In his articles, Freudenthal took Van Hiele's theory from its context of geometry and abstracted it to a procedure of logical analysis in gaining insight in (the levels of) any mathematical subject that was to be taught.

In two articles from 1962 and 1963 he took a crucial step when, for the first time, he used two important pillars of his didactical ideas which were closely related i.e. the so-called 'anti-didactical inversion' and the principle of 'guided reinvention'. These expressions summarize what the level theory implied for him: starting mathematics education in a concrete way, providing students with the opportunity to experiment and to investigate and teaching students at a level they can grasp without making concessions to the 'exactness' of the subject. Above all, mathematics should not be identified with the result, but analysed and taught as an activity. In this Freudenthal's intuitionistic background can be recognised: mathematics was considered a mental creation and this assumption should also be the starting point in education.

In the late fifties, when the 1958-curriculum was barely in working order, the urge for a drastic change in course was felt. In Europe, influenced by the American ‘New Math’, the call to introduce modern mathematics in the classroom, was also heard. Although the 1958-curriculum was better organized and easier to work with, it hardly contained ‘modern’ mathematics. A turbulent time had come for mathematics education. Freudenthal emerged from this period as the lonely opponent of New Math, not taken seriously by anybody – not in the least by portraying himself as such with due pride.

This picture does not seem entirely correct (see Chapter VIII). Freudenthal’s real violent criticism on New Math-initiatives mainly concerned foreign tendencies, such as those in Belgium under supervision of F. Papy. In the more conservative Dutch field of mathematics education such tendencies were never under discussion. What did happen in the Netherlands in the sixties, happened with support and collaboration of Freudenthal.

His whole life he held on to the motto: ‘better to cooperate and adjust, than to stand as an opponent outside the field of action’. Furthermore, Freudenthal was very consistent in placing himself above the current New Math-discussion by demanding not the modernization of the *contents* of mathematics education, but the modernization of mathematics education *as such*, thus emphasizing the modernization of the teaching methods. According to him, this could be attained by refresher courses for teachers in modern mathematics, such as the ones organised by the ‘CMLW’ (Commission for the modernization of the mathematics curriculum), in which Freudenthal played an active role. He was confident that a teacher with sufficient knowledge of higher and modern mathematics – besides having the due knowledge of the logical analysis of the learning process – should be able to develop the accompanying didactics and methodology himself.

Freudenthal’s point of view in the sixties was ‘introduction to the mathematical method, to genuine mathematics, useful for students with acumen’. It would be a nice incidental circumstance if school mathematics could be modern mathematics. Freudenthal hardly gave any specific proposals for subject matter.

Gradually, he discovered that his goal was not so easily reached as he originally thought. Venting his opinion in lectures and articles, and the organizing of refresher courses for teachers was not enough. He needed a bolder action.

In the late sixties an – at first rather cautious – culture change in mathematics education was noticeable. The attention gradually shifted from the mathematical contents to the child learning mathematics and to the (micro-)didactics involved. Internationally this attention for subject didactics could be seen in the growing number of international congresses and meetings bearing mathematics education as a theme. Freudenthal attended many of these congresses.

In the Netherlands the foundation of the IOWO in 1971 would prove to be an important factor in these developments. As an institute, it was unique in the Netherlands and as such a sign of the growing importance subject-related didactics received – that is, in mathematics. Apparently there was room for such an initiative

in this new didactical climate, although the foundation did not go very smoothly – mathematics narrowly escaped having to wait for a coordinating curriculum institute like all the other curriculum subjects. The influence of mathematics education as developed at the IOWO would prove to be unprecedented. In the IOWO-approach the motto ‘observing the child’ was also (and particularly there) adopted.

Though almost at retiring age, Freudenthal still took part in all the action (see Chapter IX). Relentlessly critical, but with noticeable interest for more positive action. Next to his ever growing amount of articles and lectures on mathematics education, in 1973 his book *Mathematics as an educational task* appeared in German and then in English. Here nearly all his previously published articles were combined in one voluminous work of almost 700 pages. It would not so much function as a book to be read cover-to-cover (although it was written in this way), but as an important reference book for mathematics educators. The contents did not add any spectacularly new insights to Freudenthal’s ideas as propagated in his articles and lectures, but were of great significance as evidence of his almost complete devotion to mathematics education.

Apart from these publications he also actively participated in the field of mathematics education, his activities ranging from organizing congresses and symposia to observing the practice of the classroom. As professor and head of the IOWO he became the mathematical didactician in the Netherlands, giving the institute an enormous impulse by doing so.

Filled with admiration and enthusiasm for the IOWO-work, he promoted the institute whenever he could. Almost daily he could be found at the institute. Through (but also apart from) the IOWO Freudenthal kept developing, accentuating and extending his ideas. The IOWO also provided an outlet and a sounding board for his other – almost forgotten and in the fifties and sixties hardly ever touched – interest i.e. the observation of children and their learning processes and, by sharing these experiences with others, to incorporate them into teaching.

His grandchildren were an important source of inspiration in this. In ‘Rekendidaktiek’ he already wrote about the importance of observing learning processes and the recording of ‘seemingly unimportant things, which might surprise more and give more room for thought than any profound theoretical research’. While having subjected his own sons to never-ending Socratic inquiries, he tried during their walks together, to observe his grandson Bastiaan as objectively as possible. Filling various notebooks with his observations, Freudenthal used these as illustrations in his articles and lectures. In these articles he not so much emphasized the boy’s adventures as well as the role and the power of observing small children in their own world. Especially in mathematics education it was necessary to start with small, everyday things. In doing so, he himself set an example by not considering himself, a professor, too high to stoop to the level of a toddler.

For Freudenthal, observations like these actually led to the development of theories and new premises such as the concept ‘paradigm’, a much-discussed topic in

several of his articles from the seventies. Looking for paradigms – crucial examples of reflecting on and making conscious that which you already unconsciously do or know – proved to be an important tool for him in the development of teaching. Such an example served as a starting point for generalisation and for reflection on your own actions. Finally this would lead to a leap in the learning process and with that a certain rise in the level of understanding. The ‘paradigm’ as a concept was a typical example of how Freudenthal would adopt a non-mathematical term and, in interaction with IOWO-co-workers, give it a new meaning.

Another, closely related IOWO-/Freudenthal hobbyhorse was the rich, realistic context. He had founded this idea in the fifties, but now saw it put into practice, elaborated on and applied in the development of education as took place in the IOWO. Using the IOWO as a means to come down to the essence – the practice of the classroom – as often as he wanted, it was the first time that he could accentuate, concretize and underpin his, until now, mostly theoretical ideas on mathematics education with his, so highly valued, experiences from practice. In combination with his regained enthusiasm for the child’s world, this gave him material for many lectures, articles and new books. In 1973 he described himself as ‘a mathematician who had always (and in later years exclusively) been interested in education’. In his interests and activities the scales had tipped permanently to the side of education: Freudenthal – the unapproachable professor in mathematics, the childish grandfather, the inexhaustible critic – had himself become mathematics education.

However, Freudenthal’s ideas on mathematics education did not end here. He would still work ceaselessly on the development of mathematics education and didactics up until his death in 1990. This study ends at a point where the basis of Freudenthal’s didactical ideas have largely taken a definite shape. The direction in which his work will further develop is roughly given. Finally in Chapter x, the epilogue, a glance backwards and forwards is given. Freudenthal died on an afternoon walk, on a bench in the park, where he was found by children. The circle was closed.